JOURNAL OF APPROXIMATION THEORY **52**, 132–140 (1988)

The Natural Best L_1-Approximation by Nondecreasing Functions

ROBERT HUOTARI AND DAVID LEGG,

Department of Mathematical Sciences,
Indiana University–Purdue University at Fort Wayne,
Fort Wayne, Indiana 46805, U.S.A.

AARON D. MEYEROWITZ

Department of Mathematics, The Ohio State University,
Columbus, Ohio 43210, U.S.A.

AND

DOUGLAS TOWNSEND

Department of Mathematical Sciences,
Indiana University–Purdue University at Fort Wayne,
Fort Wayne, Indiana 46805, U.S.A.

Communicated by Charles K. Chui

Received February 18, 1985; revised July 1, 1985

We construct a candidate for the natural best L_1-approximation to an integrable function, f, by elements of an L_1-closed convex proximinal set. If f is a Lebesgue integrable function on [0, 1] and the approximating set is the set of all nondecreasing functions, we show that our construction gives an extension of the known natural best L_1-approximation operator from $\bigcup_{p>1} L_p$ to L_1. In the course of doing this, we also complete the characterization, given in (Huotari, Meyerowitz, and Sheard, *J. Approx. Theory* **47** (1986), 85–91) of the set of all best L_1-approximations. Finally, in the case of isotonic approximation to a function of several variables, we extend a previous result concerning the almost everywhere convergence of the best L_p-approximations, $p>1$, to the natural best L_1-approximation.

1. INTRODUCTION

Let $(\Omega, \mathfrak{A}, \mu)$ be a finite measure space. For $1 \leqslant p < \infty$, let $L_p = L_p(\Omega, \mathfrak{A}, \mu)$ and let $L_{1+} = \bigcup_{p>1} L_p$. Suppose $f \in L_1$ and $C \subset L_1$ is a closed convex set which is *proximinal*, i.e., for any g in L_1, there is an L_1-nearest point to g in C. If $p \geqslant 1$ and f is in L_p, let $\mu_p(f, C)$ be the set of all best L_p-approximations of f in C, i.e., the set of all g in C with

$$\|f-g\|_p = \inf\{\|f-h\|_p : h \in C \cap L_p\}.$$

0021-9045/88 $3.00

If $p > 1$, it is well known that $\mu_p(f, C)$ consists of a single function, which we denote by f_p.

An element f_1 in $\mu_1(f, C)$ is called a *natural* best L_1-approximation of f in C if for each g in $\mu_1(f, C)$, $g \neq f_1$, there exists $p(g) > 1$ such that

$$\|f - f_1\|_p < \|f - g\|_p \qquad \text{for all } p \text{ in } (1, p(g)). \tag{1.1}$$

By Proposition 4 and Theorem 2 in [3], condition (1.1) is satisfied by a unique element, f_1, of $\mu_1(f, C)$, f_1 is the unique best L_1-approximation of f in C minimizing

$$\int |f - g| \ln |f - g| \, d\mu \tag{1.2}$$

among all g in $\mu_1(f, C)$, and

$$f_p \to f_1 \qquad \text{in } L_1 \text{ as } p \downarrow 1. \tag{1.3}$$

Define the operator $N_C: L_{1+} \to C$ by $N_C(f) = f_1$. In this paper, we define an operator $N_C^*: L_1 \to C$. We conjecture that $N_C^* = N_C$ on L_{1+}. In the case of isotonic approximation we show that this is so. (In a forthcoming paper we will show that it is also true if C is the set of all functions measurable with respect to an arbitrary sigma algebra.) In the case of isotonic approximation on the unit n-cube, we show that the convergence in (1.3) is pointwise almost everywhere.

2. A Candidate for $N_C(f)$, $f \in L_1$

Let f be an arbitrary element of L_1. Our goal in this section is to construct a "natural" best L_1-approximation to f in C. If N_C were continuous on L_{1+}, the fact that the set of all simple functions is dense in L_1 would make this an easy problem. The following example however, shows that N_C is not continuous.

Example 2.1. Let $\Omega = [0, 1] \subset \mathbb{R}$, $\mu =$ Lebesgue measure; let $\mathfrak{A}$ be the μ-measurable subsets of Ω and C the set of all constant functions. Let $f = I_{[0, 1/2]}$, i.e., $f(x) = 1$ if $x \in [0, \frac{1}{2}]$ and $f(x) = 0$ otherwise. For $\varepsilon > 0$, let $f^\varepsilon = I_{[0, 1/2 + \varepsilon]}$. Then $\|f - f^\varepsilon\|_1 = \varepsilon$ but $f_1 \equiv \frac{1}{2}$ while $f_1^\varepsilon \equiv 1$. The same result holds if, instead of constant functions, C consists of all nondecreasing functions in L_1.

For any functions g and h in L_1, let $g \vee h = \max(g, h)$ and $g \wedge h = \min(g, h)$, and, for nonnegative integers m and n, denote the truncations of g by $g^{\infty n} = g \wedge n$, $g^{m\infty} = g \vee (-m)$ and $g^{mn} = (g \wedge n) \vee (-m)$.

We construct our candidate for a natural best L_1-approximation to f by considering truncations of f.

For each pair (m, n) of nonnegative integers, $f^{mn} \in L_\infty$. Thus, f^{mn} has a natural best L_1-approximation, f_1^{mn}, in C. Also, if $n \geqslant k \geqslant 0$ and $0 \leqslant m \leqslant l$, then

$$f^{mn} \geqslant f^{mk} \geqslant f^{lk}.$$

Since N_C and the operators $f \to \inf \mu_1(f, C)$ and $f \to \sup \mu_1(f, C)$ are monotone (see Proposition 5 and Lemma 3 in [3]), we have

$$\sup \mu_1(f^{0\infty}, C) \geqslant f_1^{mn} \geqslant f_1^{mk} \geqslant f_1^{lk} \geqslant \inf \mu_1(f^{\infty 0}, C), \qquad \text{a.e.}$$

Thus for all integers $n > 0$,

$$\lim_{j \to \infty} f_1^{jn} = f_1^{\infty n} \qquad \text{exists a.e.} \tag{2.1}$$

and

$$\inf \mu_1(f^{\infty 0}, C) \leqslant f_1^{\infty n} \leqslant \sup \mu_1(f^{0\infty}, C) \qquad \text{a.e.} \tag{2.2}$$

It follows from (2.1) and (2.2) and the dominated convergence theorem that f_1^{jn} converges to $f_1^{\infty n}$ in L_1. By Theorem 1 in [4], $f_1^{\infty n} \in \mu_1(f^{\infty n}, C)$. Since $f_1^{mn} \geqslant f_1^{mk}$ a.e. For $n \geqslant k$, we have $\lim_{m \to \infty} f_1^{mn} \geqslant \lim_{m \to \infty} f_1^{mk}$ a.e., or,

$$f_1^{\infty n} \geqslant f_1^{\infty k} \qquad \text{a.e. for } n \geqslant k. \tag{2.3}$$

From (2.2) and (2.3) we conclude that

$$\lim_{j \to \infty} f_1^{\infty j} = f_1^* \qquad \text{exists a.e.}$$

and

$$\inf \mu_1(f^{\infty 0}, C) \leqslant f_1^* \leqslant \sup \mu_1(f^{0\infty}, C) \qquad \text{a.e.}$$

Again it follows from the dominated convergence theorem that

$$f_1^{\infty j} \to f_1^* \qquad \text{in } L_1 \text{ as } j \to \infty,$$

so Theorem 1 in [4] implies that $f_1^* \in \mu_1(f, C)$. We summarize our results in the following lemma.

LEMMA 2.2. *If $f \in L_1$, then there exists an element f_1^* of $\mu_1(f, C)$ so that* *the natural best approximations to the truncations of f converge in L_1 to f_1^*; that is*

$$\lim_{n \to \infty} \left(\lim_{m \to \infty} f_1^{mn} \right) = f_1^*.$$

We conjecture that $f_1^* = f_1$ when $f \in L_{1+}$.

3. The Natural Best Isotonic Approximation

In this section we restrict our attention to the case where $\Omega = [0, 1] \subset \mathbb{R}$, μ = Lebesgue measure, $\mathfrak{A}$ = all μ-measurable sets and $C = M$, the set of all nondecreasing functions on $[0, 1]$. If $p \geqslant 1$ and $f \in L_p$, then

$$\inf_{g \in M} \|f - g\|_p \leqslant \|f\|_p$$

(since $0 \in M$), whence $\mu_p(f, M) = \mu_p(f, M \cap L_p)$. The set $M \cap L_1$ is an L_1-closed convex lattice with $a(M \cap L_1) + b \subset M \cap L_1$ when $a \geqslant 0$, $b \in \mathbb{R}$, so the results in [3] apply. We will show that the construction in Section 2 provides an extension of N_M from L_{1+} to L_1.

We begin with a construction of $\inf \mu_1(f, M)$ and $\sup \mu_1(f, M)$. This construction is of independent interest.

For each c in $\mathbb{R}$, define

$$h_c(x) = \begin{cases} -1, & f(x) \leqslant c \\ 1, & f(x) > c, \end{cases}$$

and $k_c(x) = \int_0^x h_c(t)\,dt$. Then k_c is a continuous function of x and, for each x, $k_c(x)$ is continuous from above as a function of c. Let

$$m_c = \min_x k_c(x)$$

and

$$x_c = \max\{x : k_c(x) = m_c\}.$$

Then $x_c \leqslant x_d$ whenever $c < d$. Indeed, suppose that $x_c > x_d$ but $c < d$. Since $k_c(x_c) = k_c(x_d) + \int_{x_d}^{x_c} h_c(t)\,dt$ and $k_d(x_c) = k_d(x_d) + \int_{x_d}^{x_c} h_d(t)\,dt$, it is necessary that $\int_{x_d}^{x_c} h_c(t)\,dt \leqslant 0$ and $\int_{x_d}^{x_c} h_d(t)\,dt > 0$. Thus there exists t in $[x_d, x_c]$ such that $h_d(t) > h_c(t)$, which contradicts the definition of h_c. Thus, there exists a unique function $\underline{f}$ which satisfies the condition

$$\{x : \underline{f}(x) \leqslant c\} = [0, x_c], \qquad c \in \mathbb{R}.$$

Similarly, let

$$\bar{h}_c(x) = \begin{cases} -1, & f(x) < c \\ 1, & f(x) \geqslant c, \end{cases}$$

$$\bar{k}_c(x) = \int_x^1 \bar{h}_c(t)\, dt, \qquad M_c = \max_c \bar{k}_c(x),$$

and

$$\bar{x}_c = \min\{x : \bar{k}_c(x) = M_c\}$$

and let $\bar{f}$ be the function which satisfies the condition

$$\{x : \bar{f}(x) \geqslant c\} = [\bar{x}_c, 1], \qquad c \in \mathbb{R}.$$

THEOREM 3.1. *For $\underline{f}$ and $\bar{f}$ as defined above,*

$$\underline{f} = \inf \mu_1(f, M) \in \mu_1(f, M) \qquad \textit{and} \qquad \bar{f} = \sup \mu_1(f, M) \in \mu_1(f, M).$$

Proof. By Lemma 3 in [3], $\mu_1(f, M)$ is nonempty and contains $\inf \mu_1(f, M)$. Let $g = \inf \mu_1(f, M)$. We wish to show that $\underline{f} = g$. Since $\underline{f}(0) = g(0) = -\infty$, it is enough to show that $\underline{f} = g$ on $(0, 1]$. Suppose that $\underline{f}(x) < g(x)$ for some x in $(0, 1]$ and let $c = \underline{f}(x)$. Since g is left continuous on $(0, 1]$, $[g \leqslant c] = [0, x^*]$ for some $x^* < x_c$. Then $k_c(x^*) \geqslant k_c(x_c)$, so

$$\mu([f \leqslant c]; [x^*, x_c]) \geqslant \tfrac{1}{2}, \tag{3.1}$$

where $\mu(A; B)$ denotes the relative measure of A in B, i.e., $\mu(A; B) = \mu(A \cap B)/\mu B$. Since g is not constant at x^*, (2) in [1] gives

$$\mu([f < g]; [x^*, x_c]) \leqslant \tfrac{1}{2}. \tag{3.2}$$

Since $c < g$ on $[x^*, x_c]$, (3.1) and (3.2) show that

$$\mu([f \leqslant c]; [x^*, x_c]) = \tfrac{1}{2}$$

and

$$\mu([c < f < g]; [x^*, x_c]) = 0.$$

We now will show that there is a function $\phi \in M$ with $\|f - \phi\|_1 \leqslant \|f - g\|_1$ and $\phi > g$ on $[x^*, x_c]$, contradicting the choice of g. Let

$$\phi(x) = \begin{cases} c, & x \in [x^*, x_c] \\ g(x) & \text{otherwise.} \end{cases}$$

We have seen that almost everywhere on $[x^*, x_c]$, either $f \leqslant c \leqslant g$ (so $h_c = -1$) or $c < g \leqslant f$ (so $h_c = 1$). Thus

$$\begin{aligned} \|f-\phi\|_1 - \|f-g\|_1 &= \int_{x^*}^{x_c} (g(x)-c)\, h_c(x)\, dx \\ &= \int_{x^*}^{x_c} \int_c^{g(x)} h_c(x)\, dy\, dx \\ &= \int_c^{g(x_c)} \int_{g^{-1}(y)}^{x_c} h_c(x)\, dx\, dy \\ &= \int_c^{g(x_c)} [k_c(x_c) - k_c(g^{-1}(y))]\, dy, \end{aligned}$$

where $g^{-1}(y) = \inf\{x\colon g(x) \geqslant y\}$. By definition of x_c, the last integrand is always nonpositive so $\|f-\phi\|_1 - \|f-g\|_1 \leqslant 0$. Since $\phi < g$ on $[x^*, x_c]$, we have a contradiction, so $\underline{f} \geqslant g$.

Suppose now that $\underline{f}(x) > g(x)$ for some x in $(0, 1]$. Let $c = g(x)$ and let $[g \leqslant c] = [0, x^*]$. Then $x_c < x \leqslant x^*$ and $k_c(x_c) < k_c(x^*)$ so

$$\mu([f > c]\colon [x_c, x^*]) > \tfrac{1}{2}. \tag{3.3}$$

Either $x^* = 1$ or $0 < x^* < 1$. In the second case $g(t) > g(x^*)$ whenever $t > x^*$, so (3) in [1] implies that

$$\mu([f \leqslant g]; [x_c, x^*]) \geqslant \tfrac{1}{2}. \tag{3.4}$$

If $x^* = 1$, then (4) in [1] gives (3.4). Since $g \leqslant c$ on $[x_c, x^*]$, (3.3) and (3.4) are contradictory. Thus $\underline{f} \leqslant g$, so $\underline{f} = g$.

The demonstration that $\bar{f} = \sup \mu_1(f, M)$ is similar. This concludes the proof of Theorem 3.1.

We now recall a characterization of $\mu_1(f, M)$ which was given in [1]. In that paper $\underline{f}$ and $\bar{f}$ were defined differently than they are here, but Theorem 3.1 shows that both definitions give the same functions. Let U be the union of all maximal open intervals on which $\bar{f}$ and $\underline{f}$ are constant and unequal. In [1] it was shown that $\bar{f} = \underline{f}$ almost everywhere on $[0, 1] - U$. Define $h\colon [0, 1] \to \mathbb{R}$ by

$$h(x) = \begin{cases} 1, & f(x) \geqslant \bar{f}(x) > \underline{f}(x) \\ -1, & f(x) \leqslant \underline{f}(x) < \bar{f}(x) \\ 0, & \text{otherwise.} \end{cases}$$

Also let

$$k(x) = \int_0^x h(t)\, dt. \tag{3.5}$$

As shown in [1], $[k=0] \supset ([0,1]-U)$, $([k=0] \cap U)$ has measure zero, and for any g in M, $g \in \mu_1(f, M)$ if and only if

(i) $\underline{f} \leqslant g \leqslant \bar{f}$ on $[0, 1]$ and

(ii) g is constant on each component of $[k \neq 0]$.

The characterization in [1] was partial in that it was not shown how $\inf \mu_1(f, M)$ and $\sup \mu_1(f, M)$ depend on f. A by-product of Theorem 3.1 in the present paper is that the characterization is now complete. It also allows us to establish that N_M^* extends N_M.

The following lemma will combine with property (1.2) of the natural best L_1-approximation to give our result. For notational convenience, we denote $\overline{f^{mn}}$ by $\bar{f}^{mn}$ and $\underline{f^{mn}}$ by $\underline{f}^{mn}$ for any nonnegative integers m and n.

LEMMA 3.2. *Suppose $f \in L_1$ and $g \in \mu_1(f, M)$. Then for each pair (m, n) of nonnegative integers $g^{mn} \in \mu_1(f^{mn}, M)$.*

Proof. We first use Theorem 3.1 to describe $\underline{f}^{mn}$ and $\bar{f}^{mn}$ and then use conditions (i) and (ii) to show that g^{mn} is in $\mu_1(f^{mn}, M)$. For each $c \in \mathbb{R}$, define k^{mn}, k_c^{mn}, x_c^{mn}, $\bar{x}_c^{mn}$, and U^{mn} in the same way that k, k_c, x_c, $\bar{x}_c$, and U are defined for f. Then $x_c^{mn} = 0$ for $c < -m$, $x_c^{mn} = x_c$ for $-m \leqslant c < n$ and $x_c^{mn} = 1$ for $c \geqslant n$. Thus $\underline{f}^{mn} = (\underline{f})^{mn}$ on $(0, 1]$ $(\underline{f}^{mn}(0) = -\infty)$. Similarly, $\bar{f}^{mn} = (\bar{f})^{mn}$ on $[0, 1)$. Clearly $\underline{f}^{mn} \leqslant g^{mn} \leqslant \bar{f}^{mn}$ on $[0, 1]$, since $\underline{f} \leqslant g \leqslant \bar{f}$.

Let B be any component of $[k^{mn} \neq 0]$. To complete the proof we must show that g^{mn} is constant on B. To that end, we observe that B is completely contained in one of the sets

$$A_1 = [\bar{f} \leqslant -m] = [0, \bar{x}_{-m}], \quad A_2 = [\underline{f} \geqslant n] = [x_n, 1], \quad \text{and} \quad A_3 = (\bar{x}_{-m}, x_n).$$

In either of the first two cases g^{mn} is surely constant on B. Finally, examination of the definitions shows that $h^{mn} = h$ on $(\bar{x}_{-m}, x_n)$ while $k(\bar{x}_{-m}) = k^{mn}(\bar{x}_{-m}) = 0$, so $k^{mn} = k$ on A_3. Thus, if B is a component of $[k^{mn} \neq 0] \cap A_3$, it is also a component of $[k \neq 0]$. Since g is constant on B, so is g^{mn}.

Note that $\lim_{n \to \infty}(\lim_{m \to \infty} g^{mn}(x)) = g(x)$ and $|g^{mn}(x)| \leqslant |g(x)|$ for each x in $[0, 1]$, so $g^{mn} \to g$ in L_1.

THEOREM 3.3. *If $f \in L_{1+}$, then $f_1^* = f_1$.*

Proof. It suffices to show that

$$\int |f - f_1^*| \ln |f - f_1^*| \leqslant \int |f - g| \ln |f - g| \tag{3.6}$$

for every g in $\mu_1(f, M)$. Given g in $\mu_1(f, M)$, let $\{g^{mn}\}$ be the sequence guaranteed by Lemma 3.2. Then, for every m, $n \geqslant 0$,

$$\int |f^{mn} - f_1^{mn}| \ln |f^{mn} - f_1^{mn}| \leqslant \int |f^{mn} - g^{mn}| \ln |f^{mn} - g^{mn}|. \tag{3.7}$$

Let $m \to \infty$ and then let $n \to \infty$ in (3.7) to get (3.6). This concludes the proof.

Since the best L_1-approximation we have constructed is the natural best L_1-approximation when f is in L_{1+}, we have indeed extended the operator N_M from L_{1+} to L_1.

4. Almost Everywhere Convergence of f_p to f_1

In this section we generalize a result from [2] concerning the convergence of the best L_p-approximations, $p > 1$, to the natural best L_1-approximation by nondecreasing functions.

For $n \geqslant 1$, let Ω be the unit n-cube, $[0, 1]^n$. Let μ denote n-dimensional Lebesgue measure on Ω and let $\mathfrak{A}$ consist of the μ-measurable subsets of Ω. If $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ are elements of Ω, we write $x \leqslant y$ if $x_i \leqslant y_i$ for $1 \leqslant i \leqslant n$. A function g: $\Omega \to \mathbb{R}$ is said to be *nondecreasing* if x, y in Ω and $x \leqslant y$ imply that $g(x) \leqslant g(y)$. Let M consist of all nondecreasing functions. Let $f \in L_q$ and, for $1 < p < q \leqslant \infty$, let $\mu_p(f, M) = \{f_p\}$. In [2] it was shown that, for $f \in L_\infty$, as p decreases to one, f_p converges almost everywhere to f_1, the natural best L_1-approximation to f by elements of M. We now show that this result is also true if f is only assumed to be in L_{1+}.

Lemma 4.1. *If $\{g^k: k > 1\} \subset M$ and $g^k \to g^1$ in L_1, then $g^k \to g^1$ almost everywhere.*

Proof. Since a subsequence of $\{g^k\}$ converges to g^1 almost everywhere, $g^1 \in M$. By Theorem 1.1 in [2], g^1 is continuous almost everywhere. If Lemma 4.1 were false, there would be a point y in the interior of Ω at which g^1 is continuous but $g^k(y)$ does not converge to $g^1(y)$. Since $\{g^k(y)\}$ has a subsequential limit $d \neq g^1(y)$ and since any subsequence of $\{g^k\}$ converges in L_1 to g^1, we may suppose that $g^k(y) \to d$. The argument for $d < g^1(y)$ is similar to that for $d > g^1(g)$, so we give only the latter: Let $d^* = (d + g^1(y))/2$. Since g^1 is continuous at y, there exists a point $z > y$ such that for each x in the set

$$J = \{x: y_1 < x_1 < z_1, ..., y_n < x_n < z_n\},$$

$g^1(x) < d^*$. Since there exists K such that for each $k \geqslant K$, $g^k(y) > d^*$ and since each g^k is nondecreasing, we have

$$\int_y^z (g^k - g^1)\,dx > \int_y^z (d^* - g^1)\,dx > 0$$

for every $k \geqslant K$, a contradiction. This establishes Lemma 4.1.

THEOREM 4.2. *If* $\in L_{1+}$, *then* f_p *converges almost everywhere as* p *decreases to one to the natural best* L_1*-approximation to* f *in* M.

Proof. By Proposition 4 and Theorem 2 in [3], $f_p \to f_1$ in L_1 as p decreases to one. We may now apply Lemma 4.1.

REFERENCES

1. R. HUOTARI, A. MEYEROWITZ, AND M. SHEARD, Best monotone approximants in $L_1[0, 1]$, *J. Approx. Theory* **47** (1986), 85–91.
2. R. HUOTARI AND D. LEGG, "Monotone approximation in several variables, *J. Approx. Theory* **47** (1986), 219–227.
3. D. LANDERS AND L. ROGGE, Natural choice of L_1-approximants, *J. Approx. Theory* **33** (1981), 268–280.
4. D. LANDERS AND L. ROGGE, Continuity of best approximants, *Proc. Amer. Math. Soc.* **84** (1981), 683–689.